Sum-of-squares optimization in Julia

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June 13, 2017

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Nonnegative quadratic forms into sum of squares

(x_1, x_2, x_3) unique

$$p(x) = x^\top Q x$$

$$x_1^2 + 2x_1x_2 + 5x_2^2 + 4x_2x_3 + x_3^2 = x^\top \begin{pmatrix} 1 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix} x$$

$$p(x) \geq 0 \ \forall x \iff Q \succeq 0$$

cholesky

$$(x_1 + x_2)^2 + (2x_2 + x_3)^2 \longleftarrow x^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} x$$

Nonnegative polynomial into sum of squares

(x_1, x_2, x_3) → $p(x)$

(x_1, x_1x_2, x_2) → X

not unique → Q

$$p(x) = X^\top Q X$$

$$x_1^2 + 2x_1^2x_2 + 5x_1^2x_2^2 + 4x_1x_2^2 + x_2^2 = X^\top \begin{pmatrix} 1 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix} X$$

$$p(x) \geq 0 \ \forall x \impliedby Q \succeq 0$$

cholesky ↓

$$(x_1 + x_1x_2)^2 + (2x_1x_2 + x_2)^2 \longleftarrow X^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} X$$

When is nonnegativity equivalent to sum of squares ?

Determining whether a polynomial is nonnegative is NP-hard.

Hilbert 1888

Nonnegativity of $p(x)$ of n variables and degree $2d$ is equivalent to sum of squares in the following three cases:

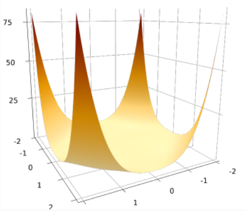
- $n = 1$: Univariate polynomials
- $2d = 2$: Quadratic polynomials
- $n = 2$, $2d = 4$: Bivariate quartics

Motzkin 1967

First explicit example:

$$x_1^4 x_2^2 + x_1^2 x_2^4 + 1 - 3x_1^2 x_2^2 \geq 0 \quad \forall x$$

but is not a sum of squares.



Sum-of-Squares cone

Nonnegative orthant $\mathbb{R}^n_+ \subset \mathbb{R}^n$

Proper and self-dual with scalar product

$$\langle a, b \rangle = b^\top a.$$

Semidefinite cone $\mathcal{S}^n_+ \subset \mathcal{S}^n$

Proper and self-dual with scalar product

$$\langle A, B \rangle = \mathrm{Tr}(BA).$$

Sum-of-Squares cone $\Sigma_{n,2d} \subset \mathbb{R}[x]_{n,2d}$

Proper and dual with scalar product

$$\langle \mu, p \rangle = \int_{\mathbb{R}^n} p(x)\mu(\mathrm{d}x).$$

is the cone of *pseudo measures*.

What is Sum-of-squares programming ?

Linear Programming

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & \langle c, x\rangle \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad \begin{aligned} \underset{y\in\mathbb{R}^n}{\text{maximize}} \quad & \langle b, y\rangle \\ \text{subject to} \quad & A^\top y \leq c \end{aligned}$$

Semidefinite Programming

$$\begin{aligned} \underset{Q\in\mathcal{S}^n}{\text{minimize}} \quad & \langle C, Q\rangle \\ \text{subject to} \quad & \langle A_i, Q\rangle = b_i \\ & Q \succeq 0 \end{aligned} \qquad \begin{aligned} \underset{y\in\mathbb{R}^n}{\text{maximize}} \quad & \langle b, y\rangle \\ \text{subject to} \quad & \sum_i A_i y_i \preceq C \end{aligned}$$

$A_i = \mathrm{Diag}(a_i), C = \mathrm{Diag}(c), Q = \mathrm{Diag}(x)$

What is Sum-of-squares programming ?

Semidefinite Programming

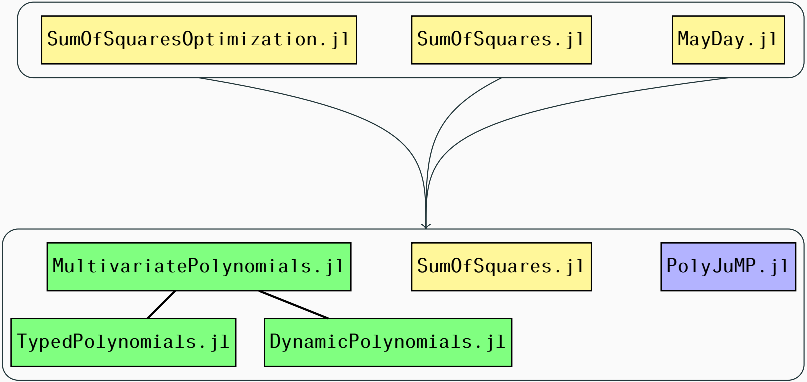
$$\begin{aligned} \underset{Q \in \mathcal{S}^n}{\text{minimize}} \quad & \langle C, Q \rangle \\ \text{subject to} \quad & \langle A_i, Q \rangle = b_i \\ & Q \succeq 0 \end{aligned} \qquad \begin{aligned} \underset{y \in \mathbb{R}^n}{\text{maximize}} \quad & \langle b, y \rangle \\ \text{subject to} \quad & \sum_i A_i y_i \preceq C \end{aligned}$$

Sum-of-squares Programming

$$\begin{aligned} \underset{p \in \mathbb{R}[x]_{n,2d}}{\text{minimize}} \quad & \langle \nu, p \rangle \\ \text{subject to} \quad & \langle \mu_i, p \rangle = b_i \\ & p \succeq 0 \end{aligned} \qquad \begin{aligned} \text{maximize} \quad & \langle b, y \rangle \\ \text{subject to} \quad & \sum_i \mu_i y_i \preceq \nu \end{aligned}$$

$(A_k)_{ij} = \langle \mu_k, x_i x_j \rangle, C_{ij} = \langle \nu, x_i x_j \rangle, p(x) = x^\top Q x$

Sum of Squares in Julia : A joint effort



Multivariate Polynomial

Choose `TypedPolynomials` or `DynamicPolynomials`:

```
using TypedPolynomials
@polyvar y # variable with name y
@polyvar x[1:2] # tuple of variables with names x1, x2
```

Build a polynomial from scratch:

```
motzkin = x^4*y^2 + x^2*y^4 + 1 - 3x^2*y^2
```

Build a vector of monomials:

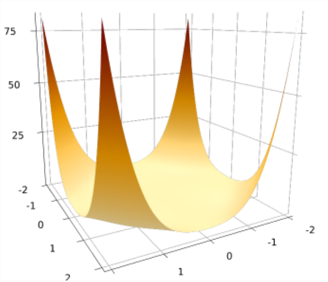
```
monomials(x, 2) # -> [x1^2, x1*x2, x2^2]
monomials(x, 0:2) # -> [x1^2, x1*x2, x2^2, x1, x2, 1]
```

Constraint

```
m = Model()
@variable m a
@constraint a * x^2 - 2x*y + a * y^2 >= 0
```

Variable

```
m = Model()
X = monomials([x, y], 0:2)
@variable m p Poly(X)
# p should be strictly positive
@constraint m p >= 1
@constraint m p * motzkin >= 0
solve(m)
```



finds $p(x) = 0.9x^2 + 0.9y^2 + 2$.

A module and and a solver

Module

PolyJuMP needs a polymodule:

```
m = Model()
setpolymodule!(m, SumOfSquares)
```

equivalent shortcut:

```
m = SOSModel()
```

2 lines version useful if multiple JuMP extensions used !

Solver

SOS variables/constraints need SDP solver, e.g. Mosek, SDPA, CSDP, SCS, ...

DSOS only need LP solver and SDSOS only need SOCP solver !

Domain constraint

Algebraic Set

Finite intersection of algebraic equalities, e.g.

```
@set x^2 == y^3 + z^3 && 2x^2 + 3y*z == x^3z^2
```

Basic semialgebraic set

Finite intersection of algebraic equalities and inequalities, e.g.

```
@set x*z >= y^2 && x + z == 1
```

```
S = @set x^2 + y^2 == 1
@constraint(m, x^2 + y <= 10,
               domain = S)
```

finds $(3 - y/6)^2 + 35/36y^2$.

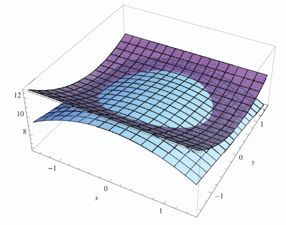


Figure 3.9 of Blekherman, Parrilo and Thomas, 2013, Semidefinite Optimization and Convex Algebraic Geometry.

SOS (resp. DSOS and SDSOS)

Variable

```
X = monomials([x, y], 0:2)
@variable m p Poly(X)
```

Variable $p(x) = X^\top QX$ where Q is semidefinite (resp. diagonally dominant, scaled diagonaly dominant).

```
@variable m p SOSPoly(X)
@variable m p DSOSPoly(X)
@variable m p SDSOSPoly(X)
```

Constraint

```
@constraint m p in SOSCone() # equivalent to p >= 0
@constraint m p in DSOSCone()
@constraint m p in SDSOSCone()
```

SOS matrix and SOS convex polynomial

Sum of square matrix

$$P(x) = \begin{bmatrix} x^2 - 2x + 2 & x \\ x & x^2 \end{bmatrix} = \begin{bmatrix} 1 & x \\ x-1 & 0 \end{bmatrix}^\top \begin{bmatrix} 1 & x \\ x-1 & 0 \end{bmatrix}$$

$$y^\top P(x) y = (y_1 + x y_2)^2 + (x-1)^2 y_1^2$$

```
@SDconstraint m [x^2-2x+2 x; x x^2] >= 0
```

Convex polynomial

Positive semidefinite hessian:

```
@SDconstraint m differentiate(p, x, 2) >= 0
```

Newton Polytope

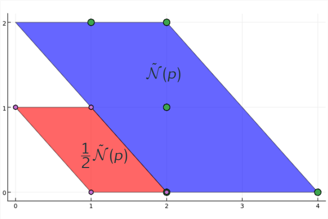
$$p(x) = X^\top Q X \qquad X = ?$$

$$p(x) = x^4 + 5x^2y^2 - 2x^2y - xy^2 + x^2$$

Default : cheap outer approx. $\tilde{\mathcal{N}}(p)$.

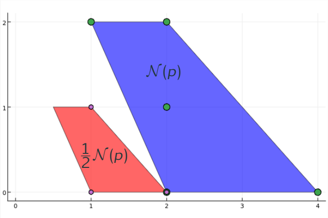
Exact newton polytope

```
@constraint(m, p >= 0,
newtonpolytope=CDDLibrary(:float))
@constraint(m, p >= 0,
newtonpolytope=CDDLibrary(:exact))
```



Sparse multipartite

```
H = differentiate(p, x, 2)
@constraint(m, y^T H y >= 0,
newtonpolytope=
  (x=>CDDLibrary(:float),
   y=>CheapOuterLibrary()))
```



Application : Polynomial optimization

Find $\min_{x \in S} p(x)$, e.g.

```
p = x^3 - x^2 + 2x*y - y^2 + y^3
S = @set x >= 0 && y >= 0 && x + y >= 1
```

SOS program:

```
m = SOSModel()
@variable m lb
@objective m Max lb
constr = @constraint m p >= lb, domain = S
```

How to recover the minimizer ? Get the dual μ and check whether it is atomic, i.e. $\mu = \sum_i \lambda_i \delta_{x_i}$.

```
AtomicMeasure(getdual(constr))
```

Atomic $\Rightarrow$ x_i global minimizers and `lb` exact minimum.

Application : Stability of Switched Systems

System $x_{k+1} = A_1 x_k$ or $x_{k+1} = A_2 x_k$. Find a common Lyapunov $V(x)$ such that $V(x) > 0$, $V(A_1 x) \leq V(x)$ and $V(A_2 x) \leq V(x)$.

```
m = SOSModel()
X = monomials(x, 2*d)
@variable m V Poly(X)
@constraint m V >= sum(x.^(2d))
@constraint m constr[i=1:2] V(x=>A[i]*x) <= V
```

How to recover an unstability certificate if it is infeasible ?

```
AtomicMeasure.(getdual(constr))
```

Atomic ⇒ μ_i occupation measure of unstable trajectory[1].

[1] See `SwitchedSystems.jl`

Future work

- Symmetry reduction.
- Different polynomial basis (Lagrange, orthogonal, ...)
- Specialized method for specific algebraic sets (e.g. hypercube) and sampling algebraic varieties.
- Modelisation with measures.
- Inclusion of decision variables in semialgebraic sets using moment relaxation.
- Non-commutative (done), hermitian, orthogonal, idempotent variables.
- Syntax for hierarchies.